Primordial Inflation and Present-Day Cosmological Constant from Extra Dimensions

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Abstract

A semiclassical gravitation model is outlined which makes use of the Casimir energy density of vacuum fluctuations in extra compactified dimensions to produce the present-day cosmological constant as $\rho_\Lambda \sim M^8/M_P^4$, where M_P is the Planck scale and M is the weak interaction scale. The model is based on $(4+D)$-dimensional gravity, with $D = 2$ extra dimensions with radius $b(t)$ curled up at the ADD length scale $b_0 = M_P/M^2 \sim 0.1$ mm. Vacuum fluctuations in the compactified space perturb b_0 very slightly, generating a small present-day cosmological constant.

The radius of the compactified dimensions is predicted to be $b_0 \approx k^{1/4} 0.09$ mm (or equivalently $M \approx 2.4$ TeV$/k^{1/8}$), where the Casimir energy density is k/b^4.

Primordial inflation of our three-dimensional space occurs as in the cosmology of the ADD model as the inflaton $b(t)$, which initially is on the order of $1/M \sim 10^{-17}$ cm, rolls down its potential to b_0.

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1 Introduction

Supernova data indicate that the energy density ρ_Λ in a present-day cosmological constant is on the order of $0.7\rho_c$, where the current critical density $\rho_c \approx (2.5 \times 10^{-3}\ \text{eV})^4$. It is intriguing that $\rho_\Lambda \sim b_0^{-4}$ where $b_0 \sim 0.1$ mm—just the length scale for compactified extra dimensions predicted by Arkani-Hamed-Dimopoulos-Dvali (ADD) type theories [1] with two extra spatial dimensions.

It is possible that this dark energy derives from vacuum fluctuations in extra compactified dimensions. We outline here a semiclassical gravitation model which makes use of this mechanism to produce the present-day cosmological constant. The model is based on $(4+D)$-dimensional gravity, with $D = 2$ extra dimensions with radius $b(t)$ curled up at the ADD length scale b_0, where the subscript "0" denotes present-day values.

The ADD model can be realized [2] in type I ten-dimensional string theory, with standard model fields naturally restricted to a 3-brane [3], while gravitons propagate in the full higher dimensional space. For $D = 2$, two of the six compactified dimensions are curled up with radius $\sim b_0$, while the remaining four are curled up with radius $\sim 1/M_I$, with the type I string scale $M_I \sim 1$ TeV. In this picture, the ADD model is formulated within a consistent quantum theory of gravity.

In addition, if supersymmetry is broken only on the 3-brane, then the bulk cosmological constant vanishes (see e.g. Ref. [4]). A single fine tuning of parameters in the potential for b can then cancel the brane tension, setting the usual four-dimensional cosmological constant to zero.

Semiclassical $(4 + D)$-dimensional gravitation—with a potential for the scale b of the extra compactified dimensions—rapidly becomes a good approximation to the string theory for energies below M_I [5]. In the semiclassical gravitation model, we will assume a potential for $b(t)$ which stabilizes $b(t_0)$ at $b_0 = M_P/M^2$ and which vanishes[1] at b_0 in the absence of the Casimir effect, where the (reduced) Planck scale $M_P = 2.4 \times 10^{18}$ GeV and the weak interaction scale $M \sim 1$ TeV. Vacuum fluctuations in the compactified space will then perturb $b(t_0)$ very slightly away from b_0, generating a small present-day cosmological constant in our three-dimensional world. This mechanism differs from previous cosmological models incorporating the Casimir effect from

[1] In other words, we assume that the 3-brane tension is exactly cancelled in the stabilization potential at $b = b_0$.

vacuum fluctuations in extra compactified dimensions (see e.g. Ref. [6]), in which the Casimir energy density in our three-dimensional world is cancelled by a bulk cosmological constant.

Primordial inflation of our three-dimensional space will occur in the model as the inflaton $b(t)$, which initially is on the order of $1/M \sim 10^{-17}$ cm, rolls down its potential to b_0 [7, 8]. Many e-folds of inflation of our 3-space can occur for sufficiently flat potentials.

We will take the spacetime metric to be $R^1 \times S^3 \times T^2$ symmetric[2] where S^3 is a 3-sphere and T^2 is a 2-torus:

$$g_{MN} = \mathrm{diag}\{1, -a^2(t)\tilde{g}_{ij}, -b^2(t)\tilde{g}_{mn}\} \tag{1}$$

where $M,\ N$ run from 0 to 5; $i,\ j$ run from 1 to 3; and $m,\ n$ run from 4 to 5. $\tilde{g}_{ij}$ is the metric of a unit 3-sphere and $\tilde{g}_{mn}$ is the metric of a unit 2-torus, with $a(t)$ the radius of physical 3-space and $b(t)$ the radius of the compactified space.

The nonzero components of the $(4+D)$-dimensional Ricci tensor are

$$\begin{aligned}
R_{00} &= -\left(3\frac{\ddot{a}}{a} + D\frac{\ddot{b}}{b}\right) \\
R_{ij} &= -\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + D\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{2}{a^2}\right) g_{ij} \\
R_{mn} &= -\left(\frac{\ddot{b}}{b} + (D-1)\frac{\dot{b}^2}{b^2} + 3\frac{\dot{a}}{a}\frac{\dot{b}}{b}\right) g_{mn}.
\end{aligned} \tag{2}$$

The generalized Einstein equations are

$$R_{MN} = 8\pi\overline{G}\left(T_{MN} - \frac{T^P{}_P}{D+2} g_{MN}\right) \tag{3}$$

where $8\pi\overline{G} = 8\pi G\mathcal{V}_0 = \mathcal{V}_0/M_P^2 = \tilde{\Omega}_D/M^{D+2}$ is the $(4+D)$-dimensional gravitational constant, $\mathcal{V}_0 = \tilde{\Omega}_2 b_0^2$ is the volume of the compactified dimensions today, $\tilde{\Omega}_D$ denotes the volume of the unit D-torus, and T_{MN} is the energy-momentum tensor. The gravitational coupling $8\pi G = 1/(b_0^2 M^4)$ is weak in the ADD picture because b_0 is much greater than the $(4+D)$-dimensional Planck length $1/M$.

[2] Our treatment through Eq. (11) parallels that of Kolb and Turner [9].

The nonzero components of the energy-momentum tensor are given by

$$\begin{aligned} T_{00} &= \rho \\ T_{ij} &= -p_a g_{ij} \\ T_{mn} &= -p_b g_{mn}. \end{aligned} \tag{4}$$

Thus $T^P{}_P = \rho - 3p_a - Dp_b$. Expressed in terms of the radii a and b, the energy density ρ, and the pressures p_a and p_b, the Einstein equations become

$$3\frac{\ddot{a}}{a} + D\frac{\ddot{b}}{b} = -\frac{8\pi\overline{G}}{D+2}\left[(D+1)\rho + 3p_a + Dp_b\right] \tag{5}$$

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + D\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{2}{a^2} = \frac{8\pi\overline{G}}{D+2}\left[\rho + (D-1)p_a - Dp_b\right] \tag{6}$$

$$\frac{\ddot{b}}{b} + (D-1)\frac{\dot{b}^2}{b^2} + 3\frac{\dot{a}}{a}\frac{\dot{b}}{b} = \frac{8\pi\overline{G}}{D+2}\left[\rho - 3p_a + 2p_b\right]. \tag{7}$$

After a few e-folds of primordial inflation of our physical 3-space, the curvature term $2/a^2$ on the left-hand side of Eq. (6) will be negligible, and we will henceforth set this term to zero.

We will be looking for solutions (neglecting matter) in which physical 3-space is inflating at the present epoch during which $b(t)$ is fixed at b_0, or in the primordial epoch just after the quantum birth of the universe during which $b(t)$ is inflating to its present value. For an inflating 3-space (without matter), $p_a = -\rho$ and the Einstein equations become

$$3\frac{\ddot{a}}{a} + D\frac{\ddot{b}}{b} = \frac{8\pi\overline{G}}{D+2}\left[-(D-2)\rho - Dp_b\right] \tag{8}$$

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + D\frac{\dot{a}}{a}\frac{\dot{b}}{b} = \frac{8\pi\overline{G}}{D+2}\left[-(D-2)\rho - Dp_b\right] \tag{9}$$

$$\frac{\ddot{b}}{b} + (D-1)\frac{\dot{b}^2}{b^2} + 3\frac{\dot{a}}{a}\frac{\dot{b}}{b} = \frac{8\pi\overline{G}}{D+2}\left[4\rho + 2p_b\right]. \tag{10}$$

The energy density and pressures on the right-hand sides of Eqs. (8)–(10) are derivable from the internal energy $U = U(a,b)$:

$$\rho = \frac{U}{\mathcal{V}}, \quad p_a = -\frac{a\partial U/\partial a}{3\mathcal{V}}, \quad p_b = -\frac{b\partial U/\partial b}{D\mathcal{V}} \tag{11}$$

where $\mathcal{V} = \Omega_3 a^3 \tilde{\Omega}_2 b^2$ is the volume of $(3 + D)$-space and Ω_3 denotes the volume of the unit 3-sphere.

We will consider a potential $V(b)$ for the radius $b(t)$ in the internal energy

$$U(a, b) = \Omega_3 a^3 M^4 V(b) \tag{12}$$

(at zero temperature) which will produce sufficient primordial inflation to solve the horizon, flatness, homogeneity, isotropy, and monopole problems, and which will stabilize b at $b_0 = M_P/M^2 \sim 0.1$ mm, with a vanishing cosmological constant. Note that if p_a is to equal $-\rho$, then U must be proportional to a^3, and that $V(b)$ is dimensionless.

The potential $V(b)$ will generate a potential $B(b)$ with the right-hand side of the Einstein equation (10) equal to $-B'(b)/b$. If $B(b)$ is sufficiently flat near $b \sim 1/M$, then many e-folds of inflation will occur in our physical 3-space as $b(t)$ rolls from $1/M$ to b_0.

Quantum fields will be periodic in the compactified space, producing a Casimir effect [6] in the compactified space and in our three-dimensional world. Adding a Casimir (C) term to the internal energy

$$U_C(a, b) = \Omega_3 a^3 \left(\frac{k}{b^4} + M^4 V(b) \right) \tag{13}$$

from vacuum fluctuations in the compactified space will perturb $b(t_0)$ very slightly away from b_0 and generate a residual present-day cosmological constant $\rho_\Lambda = k/b_0^4$. The sign and magnitude[3] of the constant k depend on the particle content and structure of the underlying quantum gravity theory. The magnitude of k may be expected to be roughly in the range 10^{-7}–10^{-3} based on the analysis of Candelas and Weinberg [6], who calculated the one-loop Casimir contribution from massless scalar and spin-$\frac{1}{2}$ particles in $(4 + D)$-dimensional gravitation with an odd number of extra dimensions D curled up near the Planck length. In their work, k is positive for a single massless real scalar field for odd dimensions $3 \leq D \leq 19$, but may be positive or negative. For our model to produce a positive present-day cosmological constant, we will need $k > 0$.

[3]A logarithmic dependence $\ln(M^2 b^2)$ can be absorbed into the definition of k without changing the conclusions below.

2 Primordial Inflation

In this section, we briefly review the cosmological results for primordial inflation of Refs. [7, 8] for the ADD model with internal energy U, and check that the Casimir terms in the Einstein equations when U is replaced by U_C do not qualitatively change the primordial cosmological picture.

The Einstein equations with the internal energy given by U in Eq. (12) take the form

$$3\frac{\ddot{a}}{a} + 2\frac{\ddot{b}}{b} = 3\dot{H} + 3H^2 + 2\dot{H}_b + 2H_b^2 = \frac{V'(b)}{4b} \tag{14}$$

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{\dot{a}}{a}\frac{\dot{b}}{b} = \dot{H} + 3H^2 + 2HH_b = \frac{V'(b)}{4b} \tag{15}$$

$$\frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} + 3\frac{\dot{a}}{a}\frac{\dot{b}}{b} = \dot{H}_b + 2H_b^2 + 3HH_b = \frac{V(b)}{b^2} - \frac{V'(b)}{4b} \equiv -\frac{B'(b)}{b} \tag{16}$$

where the Hubble parameters $H \equiv \dot{a}/a$ and $H_b \equiv \dot{b}/b$. For a vanishing present-day cosmological constant, $V'(b_0) = 0$ from Eq. (15). Eq. (16) then implies $V(b_0) = 0$ to stabilize $b(t_0)$ at b_0.

To summarize the successful phenomenology of Ref. [8]: The ADD model can produce sufficient inflation ($\gg 70$ e-folds) to solve the cosmological problems for a class of potentials $V(b)$ which satisfy

$$H^{-1} \sim H_b^{-1} \geq \frac{1}{M} \tag{17}$$

at the beginning of inflation at the quantum birth of the universe when $a \sim b \sim 1/M$, and

$$H \gg H_b, \quad \dot{H}_b \ll H^2 \tag{18}$$

during the initial stages of inflation. The correct magnitude and approximate scale invariance of density perturbations $\delta\rho/\rho = 2 \times 10^{-5}$ are created if at an intermediate stage of inflation when $b(t) \sim 10^{3/2}/M \ll b_0$, $H_b \approx H/100$. There may be a period of contraction (similar to the vacuum Kasner solutions) of our physical 3-space, but for $D = 2$, the amount of contraction of $a(t)$ is at most 7 e-folds, so the contraction phase does not invalidate the solution of the flatness problem.

Replacing U by U_C in Eq. (13) introduces Casimir terms into the Einstein equations:

$$3\dot{H} + 3H^2 + 2\dot{H}_b + 2H_b^2 = -\frac{k}{M^4 b^6} + \frac{V'(b)}{4b} \tag{19}$$

$$\dot{H} + 3H^2 + 2HH_b = -\frac{k}{M^4 b^6} + \frac{V'(b)}{4b} \tag{20}$$

$$\dot{H}_b + 2H_b^2 + 3HH_b = \frac{2k}{M^4 b^6} + \frac{V(b)}{b^2} - \frac{V'(b)}{4b} \equiv -\frac{B_C{}'(b)}{b}. \tag{21}$$

The Casimir terms do not qualitatively change the primordial inflationary period of the ADD model, since initially

$$\frac{k}{M^4 b^6} \approx kM^2 \ll M^2 \sim H^2 \sim H_b^2 \tag{22}$$

and in the intermediate stage of inflation

$$\frac{k}{M^4 b^6} \approx 10^{-9} kM^2 \ll 10^{-11} M^2 \sim H_b^2 \sim 10^{-4} H^2 \tag{23}$$

for $k \lesssim 10^{-3}$, using the estimates in Ref. [8].

3 Present-Day Cosmological Constant

In the present epoch, the internal dimensions have a fixed radius $b(t_0) \gg 1/M$ and $H_b = 0$. Without the Casimir terms, the static solution for $b(t_0)$ requires $V(b_0) = 0 = V'(b_0)$. In our model, vacuum fluctuations in the compactified space perturb b_0 very slightly to $\tilde{b}_0$, producing a small cosmological constant in our three-dimensional world. We assume that the potential $V(b)$ is independent of the Casimir effect, so that $V(b_0)$ and $V'(b_0)$ still equal zero.

The Einstein equations with Casimir contributions for an inflating 3-space now take the form

$$3H_0^2 = -\frac{k}{M^4 \tilde{b}_0^6} + \frac{V'\left(\tilde{b}_0\right)}{4\tilde{b}_0} \tag{24}$$

$$0 = \frac{2k}{M^4 \tilde{b}_0^6} + \frac{V\left(\tilde{b}_0\right)}{\tilde{b}_0^2} - \frac{V'\left(\tilde{b}_0\right)}{4\tilde{b}_0}. \tag{25}$$

Setting $\tilde{b}_0 = (1+\delta) b_0$ and solving Eq. (25) to order $\delta \sim M^4/M_P^4$ yields

$$\frac{\delta}{4} V''(b_0) + O(\delta^2) = \frac{2k}{M^4 b_0^6} \tag{26}$$

or

$$\tilde{b}_0 \approx \left(1 + \frac{8k}{M^4 b_0^6 V''(b_0)}\right) b_0 = \left(1 + O\left(\frac{kM^4}{M_P^4}\right)\right) b_0 \tag{27}$$

where $V''(b_0) \sim 1/b_0^2 = M^4/M_P^2$. Eq. (24) then predicts a present-day cosmological term

$$3H_0^2 = \frac{\delta}{4}V''(b_0) - \frac{k}{M^4 b_0^6} + O(\delta^2) = \frac{k}{M^4 b_0^6} + O(\delta^2) \tag{28}$$

or, in other words,

$$H_0^2 = \frac{8\pi G}{3}\rho_\Lambda, \quad \rho_\Lambda = \frac{k}{b_0^4} = \frac{kM^8}{M_P^4}. \tag{29}$$

This cosmological term will $\approx 0.7\rho_c$ if $b_0 \approx k^{1/4} 0.09$ mm, or equivalently if $M \approx 2.4$ TeV$/k^{1/8}$.

Note that the Casimir effect has caused the stabilized radius b_0 to *increase* slightly, yielding a positive present-day cosmological constant.

The canonically normalized "radion" field $\varphi(t) = 2M^2 b(t)$. The mass squared of the radion field is

$$m_\varphi^2 = M^4 \left.\frac{d^2V}{d\varphi^2}\right|_{\varphi_0} \sim \frac{M^4}{M_P^2} \tag{30}$$

which must be positive at $\varphi_0 = 2M^2 b_0 = 2M_P$ to have a linearly stable b_0 solution [5].

The stability properties of $B_C(b)$ in Eq. (21) are the same as of $B(b)$ in Eq. (16): the respective solutions with $b(t_0) = b_0$ and $\tilde{b}_0$ are linearly stable if the radion mass squared is positive, since the radion mass squared including the Casimir contribution

$$m_{\varphi,C}^2 = m_\varphi^2 + \frac{5kM^8}{M_P^6} \sim \frac{M^4}{M_P^2}\left(1 + \frac{5kM^4}{M_P^4}\right) \tag{31}$$

is positive if m_φ^2 is, and are globally stable if the respective potentials $B(b)$ and $B_C(b)$ are, for example, concave upward (the simplest case), since

$$B_C(b) = B(b) + \frac{k}{2M^4 b^4} + \text{const} \tag{32}$$

is concave upward if B is.

If the number of extra dimensions D is allowed to be greater than two, the Einstein equations (24) and (25) for an inflating 3-space with static $b(t)$ change to

$$3H_0^2 = -\frac{k}{M^{D+2}\tilde{b}_0^{D+4}} - \frac{D-2}{D+2}\frac{V\left(\tilde{b}_0\right)}{M^{D-2}\tilde{b}_0^D} + \frac{V'\left(\tilde{b}_0\right)}{(D+2)M^{D-2}\tilde{b}_0^{D-1}} \tag{33}$$

$$0=\frac{4k}{DM^{D+2}\tilde{b}_0^{D+4}}+\frac{4}{D+2}\frac{V\left(\tilde{b}_0\right)}{M^{D-2}\tilde{b}_0^{D}}-\frac{2}{D(D+2)}\frac{V'\left(\tilde{b}_0\right)}{M^{D-2}\tilde{b}_0^{D-1}} \tag{34}$$

but the result for the present-day cosmological constant has the same form

$$\rho_\Lambda=\frac{k}{b_0^4}=\frac{kM^{4+\frac{8}{D}}}{M_P^{\frac{8}{D}}} \tag{35}$$

where now b_0 satisfies $b_0^D M^{D+2}=M_P^2$. Thus ρ_Λ has the right parametric dependence M^8/M_P^4 only for $D=2$.

4 Conclusion

The cosmological picture presented here joins smoothly onto the primordial inflation and big-bang cosmological pictures: The quantum birth of the universe begins with a and $b\sim 1/M$. Many ($\gg 70$) e-folds of primordial inflation occur as the inflaton $b(t)$ rolls down its potential to $\tilde{b}_0$. $b(t)$ then undergoes damped oscillations about $\tilde{b}_0$, heating the universe up to a temperature T above the temperature for big-bang nucleosynthesis (BBN) and creating essentially all the matter and energy we see today. (See Refs. [7] and [10] for two differing views on the maximum value of T, above which the evolution of the universe in ADD-type theories cannot be described by the radiation-dominated Friedmann-Robertson-Walker model.) At this point, the universe evolves according to the standard big-bang picture, expanding and cooling, with a fixed small cosmological constant $\rho_\Lambda=k/b_0^4\approx(2.3\times10^{-3}\text{ eV})^4$.

This dark energy density is much less than the BBN energy density $\sim(1\text{ MeV})^4$ and plays a role in the evolution of the universe only recently, long after the equality of energy density $\sim(1\text{ eV})^4$ in matter and radiation. The radius $b(t)$ of the compactified space has not changed since well before BBN.

Finally we note that if the stabilization potential $V(b)$ vanishes at its global minimum, the resolution of the cosmic coincidences of Ref. [11] is naturally realized in the Casimir effect since parametrically $\rho_\Lambda\sim M^8/M_P^4$.

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